Inhomogeneous dielectric waveguides: a uniform asymptotic theory

1980 J. Phys. A: Math. Gen. 13 347

(http://iopscience.iop.org/0305-4470/13/1/034)

J. Phys. A: Math. Gen., 13 (1980) 347–360. Printed in Great Britain

Inhomogeneous dielectric waveguides: a uniform asymptotic theory

J M Arnold

Department of Electrical and Electronic Engineering, University of Nottingham, Nottingham, UK

Received 28 November 1978, in final form 4 June 1979

Abstract. The application of the uniform theory of differential equations with two turning points is considered in relation to the modal eigenvalue problem for an inhomogeneous dielectric waveguide (optical fibre). Expressions are deduced for the leading asymptotic order term for the eigenvalue for modes not close to cut-off.

1. Introduction

The study of propagation in inhomogeneous dielectric waveguides has recently achieved a position of some importance due to the modern development of communication systems using these devices as transmission media. Of particular importance is the theoretical prediction of the group velocity of each individual waveguide mode; quite small differences between these velocities may cause severe distortion of a signal propagating along the waveguide. Previous attempts to analyse this problem have centred around the JWKB theory, very familiar in both wave optics and quantum mechanics (Gloge and Marcatili 1973). However, it is equally well known that these methods cannot be applied in the vicinity of certain points called turning points or caustics. These points divide the coordinate domain on which the wave problem is defined into disjoint regions, and one may pass from one region to another only by the use of special connection formulae (Fröman and Fröman 1965) or by matching expansions which are uniform about these points (Langer 1934).

In problems originating in waveguide studies, however, this procedure has met with only limited success, for the principal reasons that the defining differential equation is singular in the domain of definition, and this domain is finite, requiring the application of boundary conditions at the end-points. This method, of piecewise asymptotic matching, was attempted by Kurtz and Streifer (1969), but has not ultimately led to a systematic theory of waveguide propagation.

Here we shall approach this problem from the point of view of the uniform asymptotic theory of differential equations. In these methods one tries to find an asymptotic approximation to the original problem which is valid *everywhere* on its domain of definition. To apply boundary conditions at finite end-points, one then approximates further by *deriving* the JWKB or Langer approximations *from* this uniform representation. This has the great advantage of allowing the choice of different

0305-4470/80/01347+14$01.00

(non-uniform) approximating schemes to be chosen to match the parameters of the problem in a completely systematic and highly flexible manner.

The problem we are going to consider here is defined by the differential equation

$$\mathrm{d}^2\phi/\mathrm{d}\rho^2 + (U^2 - V^2 f + \mu/\rho^2)\phi = 0, \tag{1.1}$$

where U^2 is the eigenvalue, V is a large parameter, f is a known function, $\mu = \frac{1}{4} - m^2$, m is an integer, and ϕ is a function which represents the field variation across the core of the waveguide. Boundary conditions

$$\rho = 0: \qquad \phi \sim \rho^{m+1/2} \tag{1.2a}$$

and

$$\rho = 1: \qquad \mathrm{d}\phi/\mathrm{d}\rho = K\phi \tag{1.2b}$$

have to be applied (K is a constant depending on U and V). These equations are considered in more detail in § 2. In addition, the function f satisfies the following conditions:

(i) $f = 0$ when $\rho = 0$; (1.3a)
(ii) $f = 1$ when $\rho = 1$; (1.3b)
(iii) f is an analytic function of ρ^2; (1.3c)
(iv) $U^2 - V^2 f$ has one zero in $0 \leqslant \rho \leqslant 1$; (1.3$d$)
(v) $U^2 - V^2 f + \mu/\rho^2$ has two zeros in $0 \leqslant \rho \leqslant 1$. (1.3$e$)

These conditions are all fulfilled in practice. These assumptions about f ensure that the following classification holds for the differential equation in $0 \leqslant \rho \leqslant 1$:

(i) it has two turning points;
(ii) it has a regular singularity at the origin;
(iii) its domain of definition is finite.

The methods of Lynn and Keller (1970) could be used here if $\mu = 0$ (no singularity). The eigenvalue problem on a finite domain was considered by Anyanwu and Keller (1975), but the generality of this treatment obscures the fact that a great deal more work has to be performed actually to secure an approximation for the eigenvalue. In addition, the existence of finite boundaries has the effect of permitting different types of expansion for U^2 according to whether $U^2 \ll V^2$ or $U^2 \sim V^2$. The latter case we shall consider in the following paper, as it is considerably more complex. In that case, the conditions on f entail the confluence of the boundary $\rho = 1$ and a turning point, or alternatively the confluence of both turning points, and the detailed calculations can be quite complicated. In this paper we shall formulate the uniform approximation problem and solve the eigenvalue problem for the case $U^2 \nsim V^2$. As we have said, it is necessary to treat the singularity of the differential equation, a case not covered by Lynn and Keller, and to apply boundary conditions at *finite* end-points.

The principal difficulty encountered in attempting to apply formal treatments such as those referred to in the previous paragraph is that one finds, when all the calculations have been carried through, that the condition which results for determination of the eigenvalue is strongly implicit; the eigenvalue appears as a parameter quite deeply imbedded in the arguments of various transcendental functions, and means must still be found by which an explicit condition for the eigenvalue can be obtained. This is the problem to which this paper and the following one are addressed. A direct approach to the formal implicit equations rapidly fails under the mounting complexity of the calculations which ensue, and one of the objects of this study is to consider an alternative formulation, in which specific hypotheses are made with regard to the eigenvalue *ab initio*, thereby effectively separating the dependence of the solutions on various parameters. It is, perhaps, worth noting here that this particular problem assumes major proportions only when *finite* boundaries exist. In that case, one requires two linearly independent solutions which must be correctly normalised in order to apply boundary conditions. This situation is exceptional in quantum mechanics, but normal in waveguides, and this accounts for the lack of any adequate treatment of such difficulties in the literature.

The present study, then, is to be regarded as an attempt to determine what exactly is involved in the application of uniform asymptotic methods to the problem of determining the eigenvalues of equation (1.1) under realistic conditions appropriate to its physical context.

Although the derived eigenvalue result, equation (4.11), bears a marked resemblance to the JWKB result, it is not identical to it. In addition, the equations by which higher-order approximations can be calculated are included here implicitly in equations (3.15) and (3.16), and the analysis can be developed in a relatively straightforward fashion from the foundations laid here, although the complexity of the calculations is a serious difficulty.

2. Differential equation

Here the differential equation (1.1) and its boundary conditions will be derived briefly.

The modes of a dielectric waveguide are most conveniently discussed by using the scalar approximation (Gloge 1971, Kurtz and Streifer 1969, Kirchhoff 1973). We suppose that the waveguide is composed of a cylindrical region (the core) surrounded by an infinite homogeneous medium (the cladding). The refractive index of the core is supposed to vary smoothly such that

$$n^2 = n_0^2 - (n_0^2 - n_2^2) f[(r/a)^2], \tag{2.1}$$

where, if r is the radius and a the core diameter,

$$n_0^2 = n^2 \qquad (r=0), \tag{2.2a}$$

$$n_2^2 = n^2 \qquad (r=a), \tag{2.2b}$$

and f satisfies the five conditions stated in § 1. We also suppose that the refractive index in the outer region is everywhere n_2. Thus, at $r=a$, the refractive index is continuous but its derivative is not. The scalar approximation results from the assumption that

$$n_0^2 - n_2^2 \ll n_0^2. \tag{2.3}$$

In that case, the fields of the waveguide modes are linearly polarised, and the electric field may be written, for a particular mode, as

$$\boldsymbol{E} = E_x \boldsymbol{u}_x \, \mathrm{e}^{\mathrm{i}m\theta} \, \mathrm{e}^{\mathrm{i}\beta z}, \tag{2.4}$$

where E_x is a function of r alone, and $\boldsymbol{u}_x$ is a unit vector normal to the waveguide axis (z axis). θ is the polar angle measured from the x axis. Thus, in polar coordinates (r, θ, z), E_x satisfies

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}E_x\right) + \left(n^2k^2 - \beta^2 - \frac{m^2}{r^2}\right)E_x = 0, \tag{2.5}$$

which is the wave equation for an inhomogeneous region, valid under the scalar approximation.

The scalar approximation has been discussed many times in the literature, and is a direct consequence of the physical fact that the refractive index does not change appreciably over the scale of a wavelength. Under these circumstances the ray bundle which constitutes a mode is paraxial to a good approximation, and the polarisation state of the mode is essentially linear. This is, nevertheless, an approximation, in which two nearly degenerate *vector* modes (EH and HE hybrids) are combined in such a manner as to preserve linear polarisation over very large distances (Gloge 1971). It is a very good approximation under the circumstances in question here.

By making the substitutions

$$E_x = r^{-1/2}\phi, \tag{2.6a}$$

$$\rho = r/a, \tag{2.6b}$$

$$(n_0^2k^2-\beta^2)a^2 = U^2, \tag{2.6c}$$

$$(n_0^2k^2-n_2^2k^2)a^2 = \mathrm{V}^2, \tag{2.6d}$$

$$\mu = \tfrac{1}{4}-m^2, \tag{2.6e}$$

$$n^2 = n_0^2-(n_0^2-n_2^2)f, \tag{2.6f}$$

(2.5) transforms to

$$\mathrm{d}^2\phi/\mathrm{d}\rho^2+(U^2-V^2f+\mu/\rho^2)\phi=0 \tag{2.7}$$

as stated in § 1.

In the outer region, the function ϕ satisfies

$$\mathrm{d}^2\phi/\mathrm{d}\rho^2-(W^2-\mu/\rho^2)\phi=0 \qquad (\rho\geqslant 1) \tag{2.8}$$

where

$$W^2 = (\beta^2-n_2^2k^2)a^2 \tag{2.9a}$$

$$= V^2-U^2. \tag{2.9b}$$

The solution of (2.8) is

$$\phi = \rho^{1/2}\mathrm{K}_m(W\rho) \qquad (\rho\geqslant 1), \tag{2.10}$$

where $\mathrm{K}_m(X)$ is the modified Hankel function (Abramowitz and Stegun 1965). Since ϕ and $\mathrm{d}\phi/\mathrm{d}\rho$ must be continuous at $\rho=1$, we must have

$$\mathrm{d}\phi/\mathrm{d}\rho = K\phi \qquad (\rho=1), \tag{2.11}$$

$$K = \tfrac{1}{2}+W\,\mathrm{K}_m'(W)/\mathrm{K}_m(W), \tag{2.12}$$

where ϕ is the solution of equation (2.7). The boundary condition at $\rho=0$ is that E_x must behave like ρ^m there, so

$$\phi \sim \rho^{m+1/2}, \qquad \rho\to 0, \tag{2.13}$$

is the condition for ϕ. (Alternatively we may say that $\rho^{-1/2}\phi$ must be regular at $\rho=0$.)

We note also that the substitution (2.9*b*) takes (2.7) into

$$\mathrm{d}^2\phi/\mathrm{d}\rho^2+(V^2Q^2-W^2+\mu/\rho^2)\phi, \tag{2.14}$$

where

$$Q^2 = 1 - f. \tag{2.15}$$

This equation is much more useful than (2.7) when $W^2 \ll V^2$.

3. Uniform approximation of ϕ

The principle we adopt here is to try to find a Liouville transform (Olver 1974) which takes (2.7) into a suitable equation which can be solved approximately.† Since the original equation has two turning points, we would try to obtain a related equation having two turning points also. Furthermore, the singularity at $\rho = 0$ must be reflected in the new equation. We do this by applying the Liouville transformation

$$\phi = (\mathrm{d}z/\mathrm{d}\rho)^{-1/2}\Phi, \tag{3.1a}$$

$$(z_1^2 - z^2)(\mathrm{d}z/\mathrm{d}\rho)^2 = U^2/V^2 - f. \tag{3.1b}$$

It is necessary that, in order to obtain uniform approximations later, the transformation from z to ρ (3.1b) should be analytic in both directions. If f is analytic, then this is ensured by making the zeros of both sides of (3.1b) coincide. Thus if

$$U^2/V^2 - f = 0 \qquad \text{when } \rho = \rho_1,$$

then $\rho \to \rho_1$ should imply $z \to z_1$. Integrating (3.1b) we obtain

$$\int_0^{z_1} (z_1^2 - z^2)^{1/2}\,\mathrm{d}z = \int_0^{\rho_1} \left(\frac{U^2}{V^2} - f\right)^{1/2} \mathrm{d}\rho. \tag{3.2}$$

The lower limits are set to zero to ensure that $z \to 0$ as $\rho \to 0$. The left-hand side may be integrated explicitly to give

$$\frac{\pi}{4} z_1^2 = \int_0^{\rho_1} \left(\frac{U^2}{V^2} - f\right)^{1/2} \mathrm{d}\rho. \tag{3.3}$$

Applying this transformation to the differential equation we find that Φ must satisfy

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} + \left[V^2(z_1^2 - z^2) + \frac{\mu}{\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^2 + h\right]\Phi, \tag{3.4}$$

and

$$h = -\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^{1/2} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^{-1/2} \tag{3.5}$$

(see Olver (1974) for the general properties of Liouville transforms).

To treat the singular term we note that, near $\rho = 0$, equation (3.1b) implies that, if $z \to 0$ as $\rho \to 0$,

$$z_1^2\,(\mathrm{d}z/\mathrm{d}\rho)^2 \sim U^2/V^2 + \mathrm{O}(\rho^2), \tag{3.6}$$

† Hashimoto (1976 *IEEE Trans. Micro. Theory and Tech.* *MTT*-24, pp 559–66) has used the Liouville transform in a perturbation method for slab waveguides, but this analysis is expressedly *not* asymptotic and is quite different to the methods described here.

and it is easily found from this that

$$\frac{\mu}{\rho^2}\left(\frac{d\rho}{dz}\right)^2 \sim \frac{\mu}{z^2} + O(1). \tag{3.7}$$

Thus (3.4) becomes

$$d^2\Phi/dz^2 + [V^2(z_1^2 - z^2) + \mu/z^2 + \epsilon]\Phi = 0 \tag{3.8}$$

where

$$h + O(1) = \epsilon. \tag{3.9}$$

Now it is further easily proved that, if z and ρ are analytic functions of each other, ϵ is an analytic function of z for all $0 \leqslant \rho \leqslant 1$. In that case, one may show that the differential equation (3.8) has the asymptotic solution, as $V \to \infty$,

$$\Phi \sim \Phi_0, \tag{3.10}$$

where

$$d^2\Phi_0/dz^2 + [V^2(z_1^2 - z^2) + \mu/z^2]\Phi_0 = 0 \tag{3.11}$$

(i.e. that ϵ may be neglected as $V \to \infty$). To do this we try to solve (3.8) in the form

$$\Phi = A\Phi_0 + (B/V^2)d\Phi_0/dz. \tag{3.12}$$

By substituting (3.12) in (3.8) and using (3.11), we obtain the pair of differential equations

$$\frac{dA}{dz} = \frac{1}{2V^2}\left(\frac{d^2B}{dz^2} + \epsilon B\right), \tag{3.13}$$

$$d[(z_1^2 - z^2)^{1/2}B]/dz = \tfrac{1}{2}(z_1^2 - z^2)^{-1/2}(d^2A/dz^2 + \epsilon A), \tag{3.14}$$

which have the integrals

$$A = A_0 - \frac{1}{2V^2}\int_z^{z_1}\left(\frac{d^2B}{dz'^2} + \epsilon B\right)dz', \tag{3.15}$$

$$B = -\frac{A_0}{2(z_1^2 - z^2)^{1/2}}\int_z^{z_1}\left(\frac{d^2A}{dz'^2} + \epsilon A\right)\frac{dz'}{(z_1^2 - z'^2)^{1/2}}. \tag{3.16}$$

A_0 is a constant which may be set to unity because of the homogeneity of the equations. It is now apparent that, as $V \to \infty$,

$$A \to A_0 = 1 \tag{3.17a}$$

and

$$B \to \frac{A_0}{2(z_1^2 - z^2)^{1/2}}\int_z^{z_1}\frac{\epsilon}{(z_1^2 - z'^2)^{1/2}}dz'. \tag{3.17b}$$

The form of ϵ is rather complicated, making further calculation rather difficult; however, equations (3.13) to (3.16) are well suited to numerical computation, their solutions being analytic functions of z. Nevertheless, we have done enough to show that equation (3.10) is a valid asymptotic form for Φ as $V \to \infty$, which is all we need.

Finally, we may approximate ϕ by

$$\phi \sim (\mathrm{d}z/\mathrm{d}\rho)^{-1/2}\Phi_0, \qquad V \to \infty, \tag{3.18}$$

as our final asymptotic approximation, uniform on $0 \leq \rho \leq 1$. We must now look more closely at Φ_0.

Φ_0 is a solution of (3.11); with the substitutions

$$u^2 = Vz^2, \tag{3.19a}$$

$$2(2\nu + m + 1) = Vz_1^2, \tag{3.19b}$$

this becomes

$$\mathrm{d}^2\Phi_0/\mathrm{d}u^2 + [2(2\nu + m + 1) - u^2 + \mu/u^2]\Phi_0 = 0, \tag{3.20}$$

and this is recognisable as Laguerre's differential equation. Its solutions are related to the extended Laguerre functions (Arnold 1977), the confluent hypergeometric functions (Buchholz 1961, Slater 1960) or the Whitaker functions (Whitaker and Watson 1965). The first of these forms is most suitable for our application. We write

$$\Phi_0 = u^{(m+1/2)/2}\, \mathrm{e}^{-u/2}\, L_\nu^{(m)}(u), \tag{3.21}$$

or, in terms of z,

$$\Phi_0 = z^{m+1/2} \exp(-Vz^2/2)\, L_\nu^{(m)}(Vz^2) \tag{3.22}$$

(an unimportant constant $V^{(m+1/2)/2}$ has been dropped). A contour integral exists for $L_\nu^{(m)}(u)$ (Arnold 1977†):

$$2^m\, \mathrm{e}^{-u/2} L_\nu^{(m)}(u) = (e^{-\nu\pi \mathrm{i}} W_1 + \mathrm{e}^{\nu\pi \mathrm{i}} W_2)/2\pi \mathrm{i} \tag{3.23}$$

with, for $j = 1, 2$,

$$W_j = \int_{C_j} \left(\frac{1+s}{1-s}\right)^{\nu+(m+1)/2} (1-s^2)^{(m-1)/2}\, \mathrm{e}^{-su/2}\, \mathrm{d}s, \tag{3.24}$$

where C_1 passes from $s = -1$ to $s = \infty\, \mathrm{e}^{\mathrm{i}\sigma} (\sigma > 0)$ and C_2 is the image of C_1 in the real axis, in the opposite direction. The function $L_\nu^{(m)}(u)$ reduces to the Laguerre polynomial $\mathrm{L}_q^{(m)}(u)$ when $\nu = q$, an integer.

4. Approximation for $U^2 \neq V^2$

Having now obtained an approximate solution for the original differential equation, we turn our attention to the calculation of the eigenvalue, which allows the parameter ν in (3.19*b*) to be determined. To do this, the boundary condition (2.11) must be applied at $\rho = 1$. Let us suppose that $z \to z_0$ as $\rho \to 1$. Then integrating (3.1*b*) we obtain

$$\int_{z_1}^{z_0} (z_1^2 - z^2)^{1/2}\, \mathrm{d}z = \int_{\rho_1}^{1} \left(\frac{U^2}{V^2} - f\right)^{1/2} \mathrm{d}\rho, \tag{4.1}$$

which can be used to determine z_0. The boundary condition (2.11) for ϕ at $\rho = 1$ can be converted into one for Φ at $z = z_0$ by use of (3.1*a*), and we obtain

$$\mathrm{d}\Phi/\mathrm{d}z = K_0\Phi \qquad (z = z_0) \tag{4.2}$$

† Several typographical errors occurred in this paper; see reference.

where

$$K_0 = \left(\frac{dz}{d\rho}\right)^{-1}\left[K - \left(\frac{dz}{d\rho}\right)^{1/2}\frac{d}{d\rho}\left(\frac{dz}{d\rho}\right)^{-1/2}\right] \tag{4.3a}$$

$$= \left(\frac{dz}{d\rho}\right)^{-1}\left[K + \frac{1}{2}\left(\frac{dz}{d\rho}\right)^{-1}\frac{d^2z}{d\rho^2}\right], \tag{4.3b}$$

and the derivatives are evaluated at $\rho = 1$. Now, approximating Φ by $\Phi \sim \Phi_0$, we have

$$d\Phi_0/dz \sim K_0\Phi_0 \tag{4.4}$$

as the approximate boundary condition for Φ_0. The problem now becomes that of determining ν such that (4.4) is satisfied, and with this in mind we further approximate Φ_0.

Inspection of equation (3.11) and application of the Liouville–Green approximation (Olver 1974, p 191) indicates that Φ_0 should have an approximate representation, as $V \to \infty$,

$$\Phi_0 \sim A_1\Phi_1 + A_2\Phi_2, \tag{4.5}$$

where

$$\Phi_1 = P^{-1/2}\exp\left(-V\int_{z_1}^{z} P\,dz'\right), \tag{4.6a}$$

$$\Phi_2 = P^{-1/2}\exp\left(V\int_{z_1}^{z} P\,dz'\right), \tag{4.6b}$$

$$P = (z^2 - z_1^2)^{1/2}. \tag{4.6c}$$

This approximation holds whenever $z > z_1$, and is accurate if z is not too close to z_1. A_1 and A_2 are constants which remain to be determined; this can be achieved by integrating equation (3.24) by the method of steepest descent (using the substitution $s = \tanh\theta$) and comparing the final result (using (3.22) and (3.23)) with the assumed form (4.5). This calculation is carried out in the Appendix. It turns out that

$$A_1 = A_0\cos(\nu\pi), \tag{4.7a}$$

$$A_2 = -2A_0\sin(\nu\pi), \tag{4.7b}$$

where A_0 is a constant whose value is not required because (4.4) is homogeneous. Substitution of (4.7) in (4.4) and rearrangement of the subsequent equation gives

$$2\tan(\nu\pi) \sim \frac{P^{1/2}d(P^{-1/2})/dz - VP - K_0}{P^{1/2}d(P^{-1/2})/dz + VP - K_0}\exp\left(-2V\int_{z_1}^{z_0} P\,dz'\right), \tag{4.8}$$

with the derivative and P evaluated at $z = z_0$ (except in the integral of course). The behaviour of the exponential is dominant on the right-hand side of (4.8), making it exponentially small. Therefore, asymptotically $\tan(\nu\pi) \sim 0$ and hence

$$\nu \sim q + \theta/\pi, \tag{4.9}$$

where q is an integer and

$$\theta = \tan^{-1}\left(\frac{1}{2}\frac{P^{1/2}(dP^{-1/2}/dz) - VP - K_0}{P^{1/2}(dP^{-1/2}/dz) + VP - K_0}\exp\left(-2V\int_{z_1}^{z_0} P\,dz'\right)\right). \tag{4.10}$$

Then, using equations (3.3) and (3.19*b*), we find with (4.9) and (4.10)

$$\int_0^{\rho_1} (U^2 - V^2 f)^{1/2}\, d\rho' \sim 2(2q+m+1)\frac{\pi}{4} + \theta. \tag{4.11}$$

This expression is the required condition which determines the eigenvalue U^2. Its dependence on U^2 is implicit, both because of the left-hand side, and because θ depends on U^2 also. If, however, f is quadratic ($f = \rho^2$) and θ is negligible, (4.11) reduces to

$$U^2/V \sim 2(2q+m+1),$$

which is explicit, and it can be conjectured that, if f is 'nearly' quadratic, and V is large, reasonable approximations can be obtained. Clearly, the problem is that of *separating* the terms ρ_1, f, V, U, and θ which appear in (4.11) and mix in a complex way. This can be facilitated by some specific hypothesis on U^2. We shall consider the *ansatz* that U^2/V tends to a limit as $V \to \infty$, which corresponds physically to selecting a definite mode and following it as V is made indefinitely large. However (4.11) is quite general, and may be used to determine U^2, under less specific conditions. For example, one might simply set $\theta = 0$, to obtain $\nu = q$, allow (4.11) to determine U^2 *implicitly* (approximately), then formally calculate θ using this approximate value of U^2. This leads to an iterative scheme for U^2. It does not result in simple asymptotic properties of the eigenvalue, however, and the formal structure of uniform approximation as described here becomes rather cumbersome. To take full advantage of this type of procedure, fairly radical modifications in the formalism are required, and these are to form the subject of the following paper.

To make further progress, we need some assumptions about f; we let

$$f = \rho^2(1+\epsilon g), \tag{4.12}$$

where ϵ is a small parameter and g is an arbitrary analytic function of ρ^2, such that $g = 0$ at $\rho = 1$. We will now evaluate

$$\lim_{V\to\infty}\left(\frac{U^2}{V}\right).$$

We shall show that this limit exists, and that at any finite V the error is $O(V^{-1})$. Referring back to the original approximate treatment of the differential equation, it will be observed that such an error is of the same order of magnitude as terms which have already been ignored to reach equation (3.11), and could not be accurately calculated.

This approximation reduces the mathematics to its essentials, but still supplies a useful result. In purely numerical methods of integrating equation (1.1), for example, one cannot proceed without an initial estimate of U^2, and the formula we deduce here is well suited to that purpose. Furthermore, as the dominant term of the eigenvalue expansion, it can be used to reduce the labour of obtaining higher-order terms by using it explicitly. The higher-order calculations are extremely involved and an adequate treatment of this problem would completely obscure the essential principles which we seek to elucidate here.

With these comments in mind, we shall proceed with the evaluation of the above-mentioned limit for the function f in equation (4.12). If

$$\chi_0 = \lim_{V\to\infty}\left(\frac{U^2}{V}\right) \tag{4.13}$$

exists and is finite, then U^2/V^2 is $O(V^{-1})$. Since $f \to 0$ as $\rho \to 0$, and is a function of ρ^2, we infer that

$$\rho_1^2 \sim O(V^{-1}) \tag{4.14}$$

is quite small over its whole range. In particular, let

$$\chi = U^2/V \sim \chi_0 + O(V^{-1}); \tag{4.15}$$

then

$$\chi/V = \rho_1^2(1+\epsilon g(\rho_1^2)) \tag{4.16}$$

must be solved for ρ_1.

Let

$$g = g_0 + g_1\rho^2 + g_2\rho^4 + \ldots, \tag{4.17a}$$

$$0 = g_0 + g_1 + g_2 + \ldots = g(1). \tag{4.17b}$$

Then

$$\rho_1^2 = (1+\epsilon g_0)^{-1}(\chi/V) + O[(\chi/V)^2]. \tag{4.18}$$

Performing the integration (4.11) by extracting the quadratic part of f we obtain (neglecting θ as exponentially small)

$$(2N)\pi/4 = V[\rho_1^2(1+\epsilon g_0)^{1/2}\pi/4 + O(\rho_1^4)], \tag{4.19}$$

with

$$N = 2q + m + 1. \tag{4.20}$$

Since $\rho_1 \to 0$ as $V \to \infty$, we can use (4.18) again to find

$$2N = \chi(1+\epsilon g_0)^{-1/2} + O(\rho_1^2),$$

and hence

$$\chi_0 = \lim_{V\to\infty}(\chi) = 2N(1+\epsilon g_0)^{1/2}. \tag{4.21}$$

Therefore

$$U^2/V = 2N(1+\epsilon g_0)^{1/2} + O(V^{-1}). \tag{4.22}$$

The error term becomes larger as N increases. The expression (4.22) is *exact* when $\epsilon = 0$ (quadratic index function $f = \rho^2$); then

$$U^2/V)|_{\epsilon=0} = 2N = 2(2q+m+1). \tag{4.23}$$

This is because the Liouville transformation is trivial. For this reason we may refer to this type of analysis as *quasi-quadratic* perturbation theory; the exact, non-quadratic, f is replaced by an approximate quadratic f when $V \to \infty$. Equation (4.22) is the final expression for the eigenvalue U^2. The integers q and m are taken as mode indices, each pair of values signifying a separate mode.

Two points emerge from this analysis. Firstly, the result $\nu = q$ is exactly what we would have obtained if the existence of the boundary had been ignored, and we had instead insisted that only the recessive solution Φ_1 be present. Thus, the effect of the cladding boundary is exponentially small.

Secondly, it should be noted that the methods we have used are not valid when μ is not O(1). This is because, if this were the case, the O(1) term in equation (3.7) would not be correct (it is actually $O(\mu)$), and it would therefore not be possible to neglect it along with ϵ in passing from (3.8) to (3.11). This is a serious disadvantage, as it restricts the modes for which the theory applies to those having azimuthal number $m \sim O(1)$. Some modes near cut-off have $m \sim O(V)$, and these cannot be adequately accounted for.

Equation (4.11) represents the general solution to the problem and (4.19) is the result of a specific hypothesis to separate the known from the unknown variables. Although the approach is perturbative, and will only hold for small ϵ, it should be noted that if ϵ were large, it is probable that more than one zero of $(U^2 - V^2 f)$ would exist in $0 \leq \rho \leq 1$ and the whole asymptotic theory would fail.

5. Conclusions

We have been exploring the possibility here of applying the uniform asymptotic theory of differential equations to the problem of determining the eigenvalues of the equation describing wave propagation in an inhomogeneous dielectric waveguide. We have shown how the singularity of the differential equation can be accounted for, if the azimuthal mode number m is small, and we have considered the effect of applying boundary conditions at the ends of a finite region $0 \leq \rho \leq 1$. The determination of the eigenvalue U^2 has been carried through for those modes not close to cut-off, and it has been seen that a perturbation theory can be constructed for quasi-quadratic index variations.

It is a general feature of uniform approximations that the eigenvalue is only determined implicitly, and some hypothesis must be introduced to facilitate its explicit evaluation by separation of the relevant variables. It is also a general feature that the eigenvalue U^2 cannot be determined beyond an O(1) error term, as such terms are specifically ignored (the approximation leading to equation (3.11)). To retain these terms requires the solution of (3.15) and (3.16), a formidable undertaking on any level other than the purely formal.

The extension of this analysis to modes near cut-off requires approximations different to those made in § 4, and this will be pursued separately. The object of this paper has been to lay the foundations for such an analysis. Further extension to modes having larger values of m is also required.

Acknowledgments

This work was supported by the Post Office Research Centre, Martlesham, Suffolk, while the author was at Queen Mary College, London.

It is a pleasure to acknowledge the encouragement of Professor L B Felsen, whose advice during the preparation of the manuscript was greatly appreciated.

Appendix

We have to evaluate the normalisation constants A_1 and A_2 in equation (4.5). This is done by evaluating the contour integrals in equation (3.24) and expressing the result in the same form as equation (4.5), thereby identifying the constants. The integration is performed by the steepest-descent method.

First, let $s=\tanh\theta$ in equation (3.24). Then

$$W_j=\int_{c_j'}(\mathrm{sech}\,\theta)^{m+1}\,\mathrm{e}^{uF(\theta)}\,\mathrm{d}\theta \qquad (j=1,2) \tag{A1}$$

and

$$F(\theta)=\tfrac{1}{2}(\theta\,\mathrm{sech}^2\,\theta_0-\tanh\theta), \tag{A2}$$

$$\cosh^2\theta_0=[u/2(2\nu+m+1)]. \tag{A3}$$

The contours c_1' and c_2' are shown in figure 1(a).

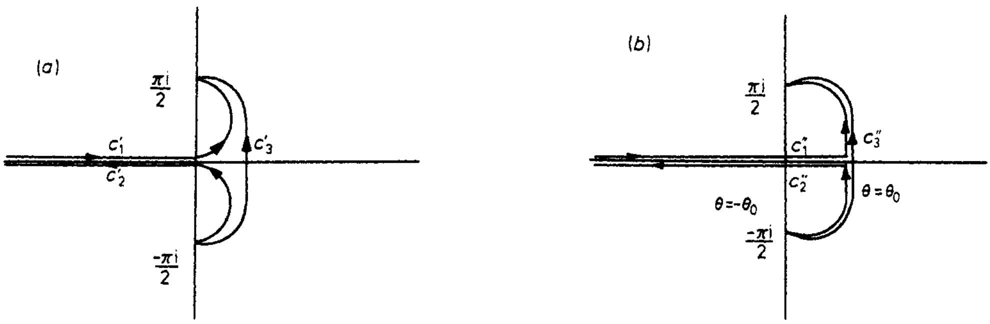

Figure 1. (a) θ-plane contours. (b) Steepest-descent contours.

The saddle points are at

$$\mathrm{d}F/\mathrm{d}\theta=0$$

i.e.

$$\theta=\pm\theta_0. \tag{A4}$$

Now

$$\begin{aligned}\mathrm{e}^{-\nu\pi\mathrm{i}}W_1+\mathrm{e}^{\nu\pi\mathrm{i}}W_2&=\cos(\nu\pi)(W_1+W_2)+\mathrm{i}\sin(\nu\pi)(W_2-W_1)\\&=\cos(\nu\pi)W_3+\mathrm{i}\sin(\nu\pi)W_4,\end{aligned} \tag{A5}$$

where

$$W_3=W_1+W_2, \qquad W_4=W_2-W_1. \tag{A6}$$

W_3 may be writen as

$$W_3=\int_{c_3'}(\mathrm{sech}^2\,\theta)^{m+1}\,\mathrm{e}^{uF(\theta)}\,\mathrm{d}\theta, \tag{A7}$$

where c_3' is shown in figure 1(a), as the integrand has no singularities between the three contours, so that

$$W_1+W_2-W_3=0 \tag{A8}$$

when W_3 is given by (A7).

Further investigation of the integrands of W_1, W_2 and W_3 shows that, on the real axis, $F(-\theta_0)$ is a maximum and $F(\theta_0)$ is a minimum. Therefore the contours c'_j can be deformed into the steepest-descent contours $c''_j\,(j=1,2,3)$ in figure 1(*b*). The steepest-descent integrals may be carried out to give (for $u\to\infty$)

$$W_1 \sim A\,(\operatorname{sech}\theta_0)^m\,(u\tanh\theta_0)^{-1/2}\exp(-uF(\theta_0)), \tag{A9}$$

$$W_2 \sim -A\,(\operatorname{sech}\theta_0)^m\,(u\tanh\theta_0)^{-1/2}\exp(-uF(\theta_0)), \tag{A10}$$

$$W_3 \sim \mathrm{i}A\,(\operatorname{sech}\theta_0)^m\,(u\tanh\theta_0)^{-1/2}\exp(uF(\theta_0)), \tag{A11}$$

where A is a constant emerging from the steepest-descent integral. Now, by equations (A3), (3.19*a*) and (3.19*b*) we have

$$z = z_1\cosh\theta_0, \tag{A12}$$

and a little algebra shows that

$$uF(\theta_0) = \frac{u}{2}(\theta_0 - \sinh\theta_0\cosh\theta_0) = -V\int_{z_1}^{z}(z'^2 - z_1^2)^{1/2}\,\mathrm{d}z'. \tag{A13}$$

Substituting (A12) and (A13) in equations (A9) to (A11) gives

$$W_1 \sim A' z^{-1/2-m} P^{-1/2}\exp\left(V\int_{z_1}^{z} P\,\mathrm{d}z'\right), \tag{A14}$$

$$W_2 \sim -A' z^{-1/2-m} P^{-1/2}\exp\left(V\int_{z_1}^{z} \mathrm{P}\,\mathrm{d}z'\right), \tag{A15}$$

$$W_3 \sim \mathrm{i}A' z^{-1/2-m} P^{-1/2}\exp\left(-V\int_{z_1}^{z} P\,\mathrm{d}z'\right), \tag{A16}$$

where

$$P = (z^2 - z_1^2)^{1/2} \tag{A17}$$

and A' is a constant.

At this point we observe that equations (A14), (A15), (A16) are not apparently consistent, since equations (A14) and (A15) indicate that

$$W_3 = W_1 + W_2 \sim 0. \tag{A18}$$

They are consistent within the asymptotics, however, because W_3 is exponentially small. By adding any multiples of W_3 to W_1 and W_2, exponentially small corrections can be made to the latter, the origin of which is the second saddle-point $\theta=\theta_0$. The coefficients of these terms cannot be determined by this method.

On using the expressions (A14)–(A17) we obtain the result

$$\Phi_0 \sim A_1\Phi_1 + A_2\Phi_2, \tag{A19}$$

where

$$A_1 = A_0[\cos(\nu\pi) + \gamma\sin(\nu\pi)], \tag{A20a}$$

$$A_2 = -2\,A_0\sin(\nu\pi), \tag{A20b}$$

where A_0 is a constant and γ is undetermined, due to the above ambiguity in the dominant solution Φ_2.

In applying the boundary conditions, it turns out that ν differs from an integer by an exponentially small amount. Hence, the second term in (A20*a*) may be neglected to obtain

$$A_1 = A_0 \cos(\nu\pi), \tag{A21a}$$

$$A_2 = -2 A_0 \sin(\nu\pi). \tag{A21b}$$

This neglect does not affect the determination to first order of the exponentially small contribution to ν implied by (4.9) and (4.10), but does affect higher corrections. These are so small as to be completely negligible, and the neglect of the undetermined term in (A20*a*) is entirely consistent.

References

Abramowitz M and Stegun I 1965 *Handbook of Mathematical Functions and Tables* (New York: Dover).
Anyanwu D U and Keller J B 1975 *Comm. Pure and App. Math.* **28** pp 753–63
Arnold J M 1977 *IEE J. MOA* **1** pp 203–8 Errata: 1978 *IEE J. MOA*, **2** p 64
Buchholz H 1961 *The Confluent Hypergeometric Function* (Berlin: Springer)
Fröman N and Fröman P O 1965 *JWKB Approximation—Contributions to the Theory* (Amsterdam: North-Holland)
Gloge D 1971 *App. Opt.* **10** pp 2252–8
Gloge D and Marcatili E A J 1973 *Bell Syst. Tech. J.* **52** pp 1563–78
Kirchhoff H 1973 *AEU* **27** pp 13–8
Kurtz C N and Streifer W 1969 *IEEE Trans. Micr. Theory and Tech.*, MTT-17, pp 250–3
Langer R E 1934 *Bull. Am. Math. Soc.* **40** pp 545–82
Lynn R Y S and Keller J B 1970 *Comm. Pure and App. Math.* **23** pp 379–408
Olver F W J 1974 *Asymptotics and Special Functions* (Academic: New York)
Slater L J 1960 *Confluent Hypergeometric Functions* (Cambridge: CUP)
Whitaker E T and Watson G N 1965 *Modern Analysis* (Cambridge: CUP)